| AP CALCULUS BC | YouTube Live Virtual Lessons | Mr. Bryan Passwater<br>Mr. Anthony Record |
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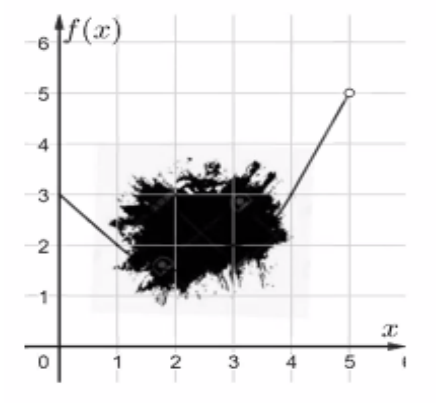
| **Topic:** From Unit 6 | **BC Integration Techniques**<br>**Free Response Question Review** | **Date:** April 14, 2020 |

| Topic Name | Topic # |
|---|---|
| Applying Properties of Definite Integrals | 6.6 |
| **Integrating Using Substitution** | 6.9 |
| **Integrating Functions Using Long Division and Completing the Square** | 6.10 |
| Integration Using **Integration by Parts *** | 6.11 |
| Using (Nonrepeating) **Linear Partial Fractions *** | 6.12 |
| **Evaluating Improper Integrals *** | 6.13 |

*** Topic is BC Only**

# 2020 FRQ Practice Problem BC1




**BC1**: For $0 \le x < 5$, the function $f$ is continuous and differentiable. A portion of the graph of $f(x)$ is obscured by a Tony Record coffee stain. It is known that $\int_0^4 f(x)dx = 8$ and $\int_4^3 f(x)dx = -3$ and $f(x)$ is linear on the intervals $(0,1)$ and $(4,5)$. For $x \ge 5$, $f(x) = \dfrac{4}{(x-a)^2}$, where $a$ is a positive real number.

(**a**) Find $\int_0^4 2xf'(x)dx$.

$$\int_0^4 2x\,f'(x)\,dx = 2\underbrace{\int_0^4 x\,f'(x)\,dx}_{\int u\,dv} \qquad \begin{array}{l} u = x \Rightarrow du = dx \\ dv = f'(x)\,dx \Rightarrow v = f(x) \end{array}$$

$$= 2\underbrace{\left[ x\,f(x)\Big|_0^4 - \int_0^4 f(x)\,dx \right]}_{uv - \int v\,du} = 2\left[\left((4\,f(4)) - ((0)\,f(0))\right) - (8)\right] = 2\left[\left((4(3)) - (0)\right) - (8)\right]$$

$$= 2\left[(12) - (8)\right] = 8$$

(**b**) Find $\int_1^2 xf(x^2-1)dx$.

$$\int_1^2 x\,f(x^2-1)\,dx = \frac{1}{2}\int_1^2 f\underbrace{(x^2-1)}_{u}\underbrace{(2x\,dx)}_{du} = \frac{1}{2}\int_0^3 f(u)\,du = \frac{1}{2}\left[\int_0^4 f(u)\,du - \int_3^4 f(u)\,du\right]$$

$$= \frac{1}{2}\left[(8) - \left(-\int_4^3 f(u)\,du\right)\right] = \frac{1}{2}\left[(8) - (-(-3))\right] = \frac{1}{2}\left[(8) - 3\right] = \frac{5}{2}$$

(**c**) Find $\int_0^{\pi/2} \cos(x) f'(\sin(x))dx$.

$$\int_0^{\pi/2} \cos(x) f'(\sin(x))dx = \int_0^{\pi/2} f'(\underbrace{\sin(x)}_{u})(\underbrace{\cos(x)dx}_{du}) = \left[f(\sin(x))\right]_0^{\pi/2} = \left[f\left(\sin\left(\frac{\pi}{2}\right)\right) - f(\sin(0))\right]$$

$$= \left[f(1) - f(0)\right] = \left[(2) - (3)\right] = -1$$

(**d**) It is known that $\int_3^{\infty} f(x)dx = 9$, find the value of $a$.

$$\int_3^{\infty} f(x)dx = \int_3^{5} f(x)dx + \int_5^{\infty} f(x)dx = \int_3^{4} f(x)dx + \int_4^{5} f(x)dx + \int_5^{\infty} f(x)dx$$

$$= (-(-3)) + \left[\frac{1}{2}(5+3)(1)\right] + \lim_{b\to\infty}\int_5^{b} f(x)dx = 7 + \lim_{b\to\infty}\int_5^{b} \frac{4}{(x-a)^2}dx = 7 + \lim_{b\to\infty}\int_5^{b} 4(x-a)^{-2}dx$$

$$= 7 + \lim_{b\to\infty}\left[-4(x-a)^{-1}\right]_5^b = 7 + \lim_{b\to\infty}\left[\frac{-4}{b-a} - \frac{-4}{5-a}\right] = 7 + \left[(0) - \frac{-4}{5-a}\right] = 7 + \frac{4}{5-a}$$

$$7 + \frac{4}{5-a} = 9 \Rightarrow \frac{4}{5-a} = 2 \Rightarrow 5 - a = 2 \Rightarrow a = 3$$

# 2020 FRQ Practice Problem BC2

**BC2**: Consider the function $g(x) = \dfrac{h(x)}{(x+2c)(x-c)}$ where $c$ is a constant with $c < 0$.

(**a**) Find $\displaystyle\int_{4c}^{7c} g(x)\,dx$ where $h(x) = 6c$.

$$\int g(x)dx = \int \frac{h(x)}{(x+2c)(x-c)}dx$$

$$= \int \frac{6c}{(x+2c)(x-c)}dx = \int \frac{-2}{(x+2c)} + \frac{2}{(x-c)}dx$$

$$= -2\ln|x+2c| + 2\ln|x-c| = 2\ln\left|\frac{x-c}{x+2c}\right|$$

$$\int_{4c}^{7c} \frac{6c}{(x+2c)(x-c)}dx = 2\left[\ln\left|\frac{x-c}{x+2c}\right|\right]_{4c}^{7c} =$$

$$= 2\left[\ln\left|\frac{6c}{9c}\right|\right] - \left[\ln\left|\frac{3c}{6c}\right|\right] = 2\left[\ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{2}\right)\right] = 2\ln\left(\frac{2/3}{1/2}\right) = 2\ln\left(\frac{4}{3}\right) = \ln\left(\frac{16}{9}\right)$$

$$\frac{6c}{(x+2c)(x-c)} = \frac{A}{(x+2c)} + \frac{B}{(x-c)}$$

$$6c = A(x-c) + B(x+2c)$$

$$x = c \Rightarrow 6c = A(0) + B(x+2c)$$

$$6c = B(3c) \Rightarrow B = 2$$

$$x = -2c \Rightarrow 6c = A(-2c-c) + B(0)$$

$$6c = A(-3c) \Rightarrow A = -2$$

(**b**) Find $\displaystyle\int_{1-2c}^{\infty} g(x)\,dx$ in terms of $c$ where $h(x) = 6c$.

$$\int_{1-2c}^{\infty} g(x)dx = \lim_{b\to\infty}\int_{1-2c}^{b} g(x)dx = \lim_{b\to\infty}\left[2\ln\left|\frac{x-c}{x+2c}\right|\right]_{1-2c}^{b}$$

$$= 2\lim_{b\to\infty}\left[\ln\left|\frac{b-c}{b+2c}\right| - \ln\left|\frac{(1-2c)-c}{(1-2c)+2c}\right|\right] = 2\lim_{b\to\infty}\left[\ln\left|\frac{b-c}{b+2c}\right| - \ln\left|\frac{(1-3c)}{1}\right|\right]$$

$$= 2\lim_{b\to\infty}\left[\ln\left|\frac{b-c}{b+2c}\right| - \ln|1-3c|\right] = 2\left[\ln(1) - \ln|1-3c|\right] = -2\ln|1-3c|$$

(**c**) Find $\int_0^{-3} g(x)\,dx$ in terms of $c$ where $h(x) = 2x + c$.

$$\int \frac{2x+c}{(x+2c)(x-c)}dx = \int \frac{1}{(x+2c)} + \frac{1}{(x-c)}dx$$

$$= \ln|x+2c| + \ln|x-c|$$

$$\int_0^{-3} \frac{2x+c}{(x+2c)(x-c)}dx = -\int_{-3}^{0} \frac{2x+c}{(x+2c)(x-c)}dx$$

$$= -\left[\ln|x+2c| + \ln|x-c|\right]_{-3}^{0}$$

$$= -\left[\left(\ln|2c| + \ln|-c|\right) - \left(\left[\ln|-3+2c| + \ln|-3-c|\right]\right)\right]$$

$$= \left[-\left(\ln|2c| + \ln|-c|\right) + \left(\ln|-3+2c| + \ln|-3-c|\right)\right]$$

$$= -\ln\left(|2c||-c|\right) + \ln\left(|-3+2c||-3-c|\right) = \ln\frac{|-3+2c||-3-c|}{|2c||-c|}$$

$$\frac{2x+c}{(x+2c)(x-c)} = \frac{A}{(x+2c)} + \frac{B}{(x-c)}$$

$$2x+c = A(x-c) + B(x+2c)$$

$$x = c \Rightarrow 3c = A(0) + B(x+2c)$$

$$3c = B(3c) \Rightarrow B = 1$$

$$x = -2c \Rightarrow -3c = A(-2c-c) + B(0)$$

$$-3c = A(-3c) \Rightarrow A = 1$$

# 2020 FRQ Practice Problem BC3

| $x$ | 1 | 2 | 3 | 4 | 6 | 8 |
|---|---|---|---|---|---|---|
| $f(x)$ | 2 | 1 | ? | 3 | 4 | 3 |
| $f'(x)$ | 4 | 2 | 1 | 2 | $-1$ | 10 |
| $g(x)$ | 1 | $-4$ | 0 | 0 | 8 | $-2$ |

**BC3**: The function $f$ is twice differentiable with domain $x \geq -10$. It is known that $f(x) > 0$ for all $x$ values in its domain and has the horizontal asymptote $y = 1$. The function $g$ is twice differentiable for all values of $x$.

(**a**) If $\int_1^3 f'(x)g(x)dx = 2\int_1^3 f(x)g'(x)dx$, find $\int_1^3 f(x)g'(x)dx$

$$\int_1^3 f'(x)g(x)dx$$

$$u = g(x) \Rightarrow du = g'(x)dx$$
$$dv = f'(x)dx \Rightarrow v = f(x)$$

$$g(x)f(x)\Big|_1^3$$
$$= g(3)f(3) - g(1)f(1)$$
$$= (0)f(3) - (1)(2) = -2$$

$$\int_1^3 f'(x)g(x)dx = g(x)f(x)\Big|_1^3 - \int_1^3 f(x)g'(x)dx$$

Let $I = \int_1^3 f(x)g'(x)dx$

$$2I = (-2) - I \Rightarrow 3I = -2 \Rightarrow I = -\frac{2}{3}$$

(**b**) If $\int_2^6 f'(g(x))g'(x)dx = -11$, find $f(-4)$.

$$\int_2^6 f'\underbrace{(g(x))}_{u}\underbrace{g'(x)dx}_{du} = \left[f(g(x))\right]_2^6 = f(8) - f(-4) = 3 - f(-4) = -11 \Rightarrow f(-4) = 14$$

# 2020 FRQ Practice Problem BC3

**The problem is restated.**

| $x$ | 1 | 2 | 3 | 4 | 6 | 8 |
|---|---|---|---|---|---|---|
| $f(x)$ | 2 | 1 | ? | 3 | 4 | 3 |
| $f'(x)$ | 4 | 2 | 1 | 2 | $-1$ | 10 |
| $g(x)$ | 1 | $-4$ | 0 | 0 | 8 | $-2$ |

**BC3**: The function $f$ is twice differentiable with domain $x \geq 0$. It is known that $f(x) > 0$ for all $x$ values in its domain and has the horizontal asymptote $y = 1$. The function $g$ is twice differentiable for all values of $x$.

(**c**) If $\int_3^\infty \frac{f'(x)}{[f(x)]^2} = 7$, find $f(3)$.

$$\int_3^\infty \frac{f'(x)}{[f(x)]^2}dx = \lim_{b\to\infty}\int_3^b \underbrace{[f(x)]}_{u}{}^{-2}\underbrace{(f'(x)dx)}_{du} = \lim_{b\to\infty}\left[-\frac{1}{f(x)}\right]_3^b = 7$$

horizontal asymptote $y = 1$
$\Rightarrow \lim_{x\to\infty} f(x) = 1$

$$= \lim_{b\to\infty}\left[\left(-\frac{1}{f(b)}\right)-\left(-\frac{1}{f(3)}\right)\right] = \left[\left(-\frac{1}{1}\right)-\left(-\frac{1}{f(3)}\right)\right] = 7$$

$$-1+\frac{1}{f(3)} = 7 \Rightarrow \frac{1}{f(3)} = 8 \Rightarrow f(3) = \frac{1}{8}$$

(**d**) Find $\int_1^2 2x^2 f''(x^3)dx$

$$\int_1^2 2x^2 f''(x^3)dx = \frac{2}{3}\int_1^2 f''\underbrace{(x^3)}_{u}\underbrace{(3x^2dx)}_{du} = \frac{2}{3}\left[f'(x^3)\right]_1^2 = \frac{2}{3}\left[f'(2^3)-f'(1^3)\right]$$

$$= \frac{2}{3}\left[f'(8)-f'(1)\right] = \frac{2}{3}[10-4] = 4$$

(**e**) Find $\int_1^2 2x^3 f''(x^2)dx$

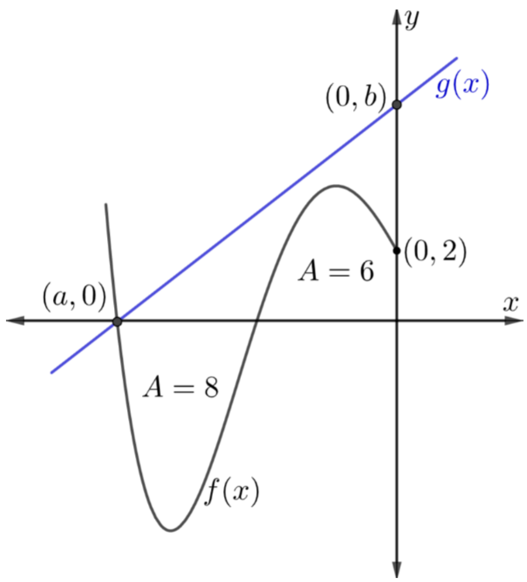
$$\int_1^2 2x^3 f''\left(x^2\right)dx = \int_1^2 \underbrace{x^2}_{u} f''\underbrace{\left(x^2\right)}_{u}\underbrace{\left(2x dx\right)}_{du} = \int_1^4 u\, f''(u)\,du$$

$$\int_1^4 u\, f''(u)\,du \qquad \begin{array}{l} w = u \Rightarrow dw = du \\ dv = f''(u)\,du \Rightarrow v = f'(u) \end{array}$$

$$= u\, f'(u)\Big|_1^4 - \int_1^4 f'(u)\,du = u\, f'(u)\Big|_1^4 - f(u)\Big|_1^4 = \left[(4)f'(4)-(1)f'(1)\right]-\left[f(4)-f(1)\right]$$

$$= \left[(4)(2)-(1)(4)\right]-\left[(3)-(2)\right] = 4-1 = 3$$

# 2020 FRQ Practice Problem BC4




**BC4**: A portion of the graphs for $f$ and $g$ are shown above where $g$ is linear with $x$ and $y$ intercepts labeled in the figure. The regions bounded by the graph of $f(x)$ and the $x$ axis have areas of 8 and 6 respectively as labeled.

(**a**) Find $\int_a^0 g(x)f'(x)dx$ in terms of $a$ and $b$.

$$g(x)=\frac{b}{a}x+b \Rightarrow g'(x)=\frac{b}{a}$$

$$u=g(x)\Rightarrow du=g'(x)dx=\frac{b}{a}dx$$

$$dv=f'(x)dx\Rightarrow v=f(x)$$

$$\int_a^0 g(x)f'(x)dx \;=\left[f(x)g(x)\right]_a^0-\frac{b}{a}\int_a^0 f(x)dx \;=\left[f(0)g(0)\right]-\left[f(a)g(a)\right]-\frac{b}{a}(-8+6)$$

$$=\left[(2)(b)\right]-\left[(0)(0)\right]-\frac{b}{a}(-2)=\left[2b\right]+\frac{2b}{a}=\frac{2b+2ab}{a}$$

(**b**) Find $\int_0^{2a}\left[f'\left(\frac{x}{2}\right)-3\right]dx$ in terms of $a$.

$$\int_0^{2a}\left[f'\left(\frac{x}{2}\right)-3\right]dx=\left[2f\left(\frac{x}{2}\right)-3x\right]_0^{2a}=\left[\left(2f(a)-6a\right)-\left(2f(0)-3(0)\right)\right]$$

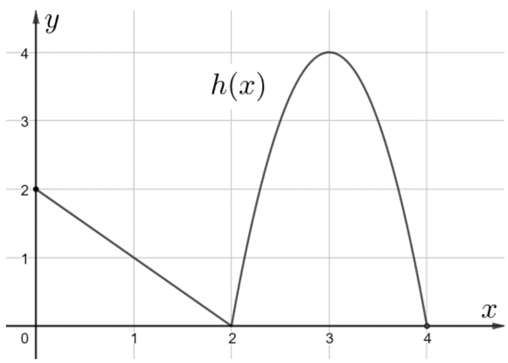
$$=\left[\left((0)-6a\right)-\left(2(2)-3(0)\right)\right]=\left[(-3a)-4\right]=-6a-4$$

(**c**) Find $\int_0^{2a} \left[f\left(\frac{x}{2}\right) - 3\right] dx$ in terms of $a$.

$$\int_0^{2a}\left[f\left(\frac{x}{2}\right)-3\right]dx=\int_0^{2a}\left[f\left(\frac{x}{2}\right)\right]dx-\int_0^{2a}[3]dx=2\int_0^{2a}\left[f\underbrace{\left(\frac{x}{2}\right)}_{u}\right]\underbrace{\left(\frac{1}{2}dx\right)}_{du}-\int_0^{2a}[3]dx$$

$$=2\int_0^{a}\left[f(u)\right]du\;-\left[3x\right]_0^{2a}=-2\int_a^{0}\left[f(u)\right]du\;-\left[6a\right]=-2(0-2)-6a=4-6a$$

# 2020 FRQ Practice Problem BC5

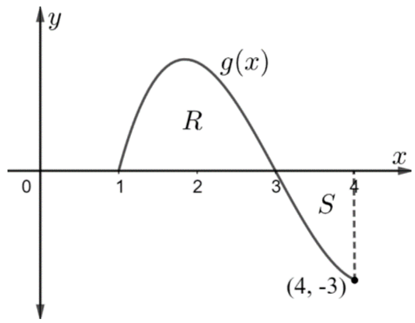



**BC5**: A portion of the continuous function $h(x)$ is given above on the interval $0 \le x \le 4$.

The function $h$ can also be defined by the equation $h(x) = 3 + \int_2^{2x} f(t)dt$.

(**a**) Find $\int_2^{6} xf'(x)\, dx$.

# 2020 FRQ Practice Problem BC6




**BC 6**: The functions $f$ and $g$ are continuous and differentiable. A portion of the graph of $g$ is given above.

The areas of the bounded regions R and S are 5 and 2 respectively. The function $f$ is defined by

$f(x) = \dfrac{1}{x^2 + k}$ where $k$ is constant.

(**a**) Find $\int x f(x) dx$ in terms of $x$ and $k$.

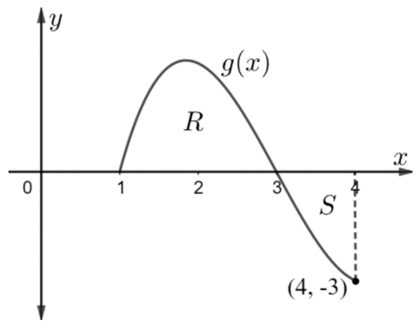
$$\int \frac{x}{x^2+k} dx = \frac{1}{2}\int \frac{1}{\underbrace{x^2+k}_{u}} \underbrace{(2xdx)}_{du} = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln\left|x^2+k\right| + C$$

(**b**) Let $k = 9$, find $\int_{\sqrt{3}}^{\infty} f(x) dx$.

$$\int_{\sqrt{3}}^{\infty} \frac{1}{x^2+9} dx = \frac{1}{3}\int_{\sqrt{3}}^{\infty} \frac{1}{\underbrace{\left(\frac{x}{3}\right)}_{u}^2 + 1} \underbrace{\left(\frac{1}{3}dx\right)}_{du} = \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_{\sqrt{3}}^{\infty} = \frac{1}{3}\lim_{b\to\infty}\left[\tan^{-1}\left(\frac{x}{3}\right)\right]_{\sqrt{3}}^{b}$$

$$= \frac{1}{3}\lim_{b\to\infty}\left[\tan^{-1}\left(\frac{b}{3}\right) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right] = \frac{1}{3}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right] = \frac{1}{3}\left[\frac{\pi}{2} - \frac{\pi}{6}\right] = \frac{1}{3}\left[\frac{\pi}{3}\right] = \frac{\pi}{9}$$

# 2020 FRQ Practice Problem BC6




**BC 6**: The functions $f$ and $g$ are continuous and differentiable. A portion of the graph of $g$ is given above.

The areas of the bounded regions R and S are 5 and 2 respectively. The function $f$ is defined by

$f(x) = \dfrac{1}{x^2 + k}$ where $k$ is constant.

(**c**) Let $k = -9$, find $\int (5x+3) f(x)dx$.

$$\int \frac{5x+3}{x^2-9}dx = \int \frac{5x+3}{(x+3)(x-3)}dx$$

$$\frac{5x+3}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$
$$5x+3 = A(x-3) + B(x+3)$$
$$x = 3 \Rightarrow 18 = 6B \Rightarrow B = 3$$
$$x = -3 \Rightarrow -12 = -6A \Rightarrow A = 2$$

$$= \int \frac{2}{(x+3)} + \frac{3}{(x-3)}dx = 2\ln|x+3| + 3\ln|x-3| + C = \ln\left|(x+3)^2(x-3)^3\right| + C$$

(**d**) Find $\int_1^4 xg'(x)dx$.

$$\int_1^4 x g'(x)dx = \left[x g(x)\right]_1^4 - \int_1^4 g(x)dx$$

$$u = x \Rightarrow du = dx$$
$$dv = g'(x)dx \Rightarrow v = g(x)$$

$$= [(4)g(4)] - [(1)g(1)] - (5-2) = [(4)(-3)] - [(0)] - (3) = -12 - 3 = -15$$